MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY.

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SIR DAVID GILL, K.C.B., F.R.S., PRESIDENT, in the Chair.

Alec Joscelyne Bamford, B.A., Assistant Astronomer to the Ceylon Government, The Observatory, Colombo, Ceylon;
Rev. Basil Staunton Batty, M.A., Bolsover Vicarage, Chesterfield, and Oxford and Cambridge Club, Pall Mall, S.W.;
George Frederick Dodwell, B.A., The Observatory, Adelaide, South Australia; and
Rev. William Francis Rigge, S.J., Professor of Physics, and Director of the Observatory, Creighton University, Omaha, Nebraska, U.S.A.,

were balloted for and duly elected Fellows of the Society.

Dr. W. de Sitter, Professor of Astronomy in the University of Leyden; and
Professor Karl Schwarzschild, Director of the Observatory, Göttingen,

were balloted for and duly elected Associates of the Society.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended:—

Thomas William Bush, Gentleman, Lullenden Farm, East Grinstead, Surrey (proposed by H. H. Turner);
John Walton Capstick, Fellow of Trinity College, Cambridge (proposed by H. F. Newall);
William Bennett Barton Freeland, J.P., Registrar of the Diocese,

Cathedral Registry Office, Chichester (proposed by F. J. W. Crowe);

R. W. Giblin, Director of the Royal Survey Department of Siam, Bangkok, Siam (proposed by Sir David Gill);

J. W. Short, Astronomical Photographer, The Observatory, Sydney, New South Wales, Australia (proposed by F. K. McClean); and

William H. Tapp, Lieut., Queen's Bays, Hounslow, Middlesex (proposed by E. A. Reeves).

Sixty-eight presents were announced as having been received since the last meeting, including, amongst others:—

F. Ristenpart, Fehlerverzeichniss zu den Sterncatalogen des 18 und 19 Jahrhunderts. Presented by the Editor of the Astronomische Nachrichten.—H. Gyldén (the late), Traité analytique des orbites absolues des huit planètes principales. Tome 2. Presented by the R. Acad. of Sciences, Stockholm.—J. Bainbridge, Canicularia [with Gravius and Ulug Beig]. 8vo. Oxford, 1648. Presented by W. H. Wesley.

On the nature of the Hydrogen Flocculi on the Sun. By G. E. Hale, Assoc. R.A.S., For. Mem. R.S.

[*From the Royal Society's Provisional Report of their Meeting, June* 17, 1909.]

Photographs of the *Hα* line in the spectrum of the solar disc, made on Mount Wilson with high dispersion, were shown on the screen. The line appears as follows:—(1) a broad dark line, differing greatly in intensity and width in different regions of the Sun. Except in eruptive or rapidly changing phenomena, the differences in width are not very marked; (2) within the boundaries of the dark line a narrow single or multiple bright line is photographed in many parts of the Sun. Sometimes the appearance resembles that of the calcium lines K_2 and K_3—*i.e.* the bright line lying on its dark background is divided into two components by a central dark line. In other regions the bright line is divided into a larger number of components, varying in width and separation.

The images of dark hydrogen flocculi, on spectroheliograph plates taken with camera slit about equal in width to *Hα*, appear to be due, in the main, to local increase in the intensity of the dark line. In some parts of the Sun, particularly those where the line is distorted, variations in the width of the line may also play an important part.

The increased intensity of the dark line is probably the result of increased absorption. Slides were shown to illustrate the fact

that prominences at the Sun's brink are frequently recorded as dark flocculi when photographed in projection against the disc.

The possible effects of anomalous dispersion were discussed, and photographs were exhibited of the same region of the Sun, taken simultaneously with light from the red and violet edges of *Ha*. The similarity of these photographs apparently indicates that anomalous dispersion is not the prime factor in producing the hydrogen flocculi. Certain minor differences suggest, however, that it may perhaps play a secondary part in modifying their form.

On Ancient Eclipses. By P. H. Cowell, M.A., F.R.S.

I have calculated the twenty-one eclipses of which a list was given by Mr. Fotheringham, *M.N.*, lxix. p. 29. In the light of Ginzel's calculations and of the constant doubt as to the exact place of observation, it was to be expected that no formulæ would render all these eclipses central at the places named by Mr. Fotheringham. The best authenticated eclipse seems to be that of + 484 at Athens, which is rendered total by both Professor Newcomb's formulæ and my own, exact centrality corresponding approximately to the mean of our formulæ. In addition to the eclipse of + 484, my formulæ satisfy the eclipses of − 187 (Rome), + 218 (Rome), + 393 (Constantinople) well; Professor Newcomb's satisfy − 393 (Chæronæa), + 447 (Chiaves) well. My formulæ satisfy + 186 (Rome) rather better than Professor Newcomb's, while for + 334 (Sicily) his formulæ satisfy rather better than mine. There is nothing noteworthy in the records except that in − 393 totality is expressly denied.

Mr. Fotheringham's last paper is most interesting, chiefly because he bases conclusions on the eclipse of − 1062 and the eclipse of Hipparchus only, with little or no stress laid on confirmation arising from other records. That is to say, that in the opinion of a historian competent to judge, these two records are intrinsically of a very high value. My belief that the eclipse of - 1062 at Babylon is trustworthy was mainly based on its agreement with several others, though I fully recognised that intrinsically it was among the best that I used. I naturally welcome Mr. Fotheringham's attitude towards this eclipse. As to the eclipse of Hipparchus, we may, it seems, fairly accept Mr. Fotheringham's verdict that at some time between the foundation of Alexandria and the death of Hipparchus an eclipse occurred, which requires to be identified with the record. As far as approximate totality at the Hellespont is concerned, Professor Newcomb's formulæ can be reconciled with the identifications − 189 March 14, − 182 October 19, − 173 October 10, and − 128 November 20. My formulæ can be reconciled with the identifications − 309 August 15, − 173 October 10, − 103 July 19. All eclipses total at the Hellespont must be large partial eclipses at Alexandria, but undoubtedly, of those above

mentioned, − 128 November 20 is the only one where the conditions are such that, if there was totality at the Hellespont, a good estimate of the magnitude at Alexandria would be four-fifths. In particular, in order to identify this eclipse with − 309 August 15, or − 103 July 19, we have to suppose that the estimated magnitude was a digit too large; if with − 173 October 10, three-quarters of a digit too small. It must, however, be borne in mind that any formulæ whatever lead to the conclusion that in lunar eclipses magnitudes are occasionally a digit wrong; and further (though I do not agree) that Professor Newcomb has stated that the probable error of a magnitude of a lunar eclipse is more than one-tenth part of the Moon's diameter (*M.N.*, lxvi. p. 472).

Note on the value of the obliquity used in the conversion of the Moon's ecliptic into equatorial coordinates. By A. M. W. Downing, D.Sc., F.R.S.

The point having been raised as to the value of the obliquity adopted in the *Nautical Almanac* calculations for the conversion of the Moon's longitude and latitude into the corresponding right ascension and declination, it may be advisable to state specifically what values have been adopted at different periods since the introduction of Hansen's Tables into the *Nautical Almanac*. This seems to be the more desirable as Hansen, at the end of his tables, gives facilities for the calculation of R.A. and Dec., thus suggesting the possibility that these subsidiary tables may have been used in the conversion throughout the whole period. The following statement will, I hope, obviate any further possible misunderstanding on the subject.

Period.	Authority for Obliquity.
1862–1874	Hansen.
1875–1900	Le Verrier.
1901 onwards	Newcomb.

1909 *June* 9.

Ephemeris of Flora near the time of Opposition in 1909.

By A. M. W. Downing, D.Sc., F.R.S.

This ephemeris is computed from Brünnow's *Tafeln der Flora*, in combination with the corrected continuation of certain of the tables published in *Monthly Notices*, vol. lxiv., No. 6.

Berlin, Midnight, 1909.		Apparent R.A.	Apparent Dec.	Log Distance from Earth.
		h m s	° ′ ″	
November	1	3 54 36·63	+ 9 18 49·2	9·95786
	2	53 47·38	16 37·7	·95660
	3	52 56·42	14 31·0	·95544
	4	52 3·85	12 29·7	·95438
	5	51 9·78	10 33·9	·95342
	6	50 14·32	8 44·3	·95256
	7	49 17·62	7 0·8	·95180
	8	48 19·75	5 24·0	·95115
	9	47 20·76	3 54·6	·95061
	10	46 20·78	2 32·5	·95018
	11	45 19·97	1 18·4	·94987
	12	44 18·38	9 0 12·0	·94966
	13	43 16·18	8 59 13·8	·94956
	14	42 13·55	58 25·1	·94958
	15	41 10·56	57 45·8	·94972
	16	40 7·34	57 15·9	·94996
	17	39 3·95	56 55·0	·95032
	18	38 0·58	56 44·6	·95079
♂	19	36 57·36	56 44·5	·95137
	20	35 54·42	56 54·8	·95207
	21	34 51·81	57 15·6	·95288
	22	33 49·61	57 46·9	·95379
	23	32 48·00	58 28·8	·95481
	24	31 47·10	8 59 22·0	·95594
	25	30 47·02	9 0 26·3	·95718
	26	29 47·83	1 41·7	·95853
	27	28 49·54	3 8·2	·95998
	28	27 52·38	4 45·8	·96152
	29	26 56·42	6 34·6	·96317
	30	26 1·75	8 34·7	·96492
December	1	25 8·43	10 45·9	·96677
	2	3 24 16·46	+ 9 13 8·1	9·96871

Berlin, Midnight, 1909.		Apparent R.A.			Apparent Dec.			Log Distance from Earth.
		h	m	s	°	′	″	
December	3	3	23	26·03	+ 9	15	41·9	9·97074
	4		22	37·27		18	27·3	·97285
	5		21	50·23		21	23·9	·97506
	6		21	4·91		24	31·8	·97734
	7		20	21·31		27	50·6	·97970
	8		19	39·65		31	20·2	·98215
	9		18	59·94		35	0·9	·98468
	10		18	22·26		38	52·5	·98728
	11		17	46·58		42	54·8	·98995
	12		17	12·84		47	7·2	·99269
	13		16	41·30		51	30·1	·99550
	14		16	11·99	9	56	3·5	9·99837
	15		15	44·91	10	0	47·0	0·00129
	16		15	20·09		5	40·7	·00428
	17		14	57·43		10	44·2	·00732
	18		14	37·10		15	57·0	·01041
	19		14	19·08		21	19·1	·01355
	20		14	3·36		26	50·1	·01673
	21		13	49·95		32	30·2	·01995
	22		13	38·79		38	18·8	·02322
	23		13	29·91		44	15·5	·02653
	24		13	23·36		50	20·4	·02987
	25		13	19·14	10	56	33·1	·03324
	26		13	17·22	11	2	53·3	·03664
	27		13	17·45		9	20·8	·04007
	28		13	19·92		15	55·3	·04353
	29		13	24·70		22	36·6	·04701
	30		13	31·75		29	24·5	·05051
	31	3	13	41·10	+ 11	36	18·9	0·05404

Magnitude at opposition, November 19 = 7·9.

Recent Observations of the Rings of Saturn, and their bearing upon some of the Phenomena of the Disappearance of the Rings in 1907. By E. E. Barnard.

In *A.N.* 4276 and 4277 announcement was made of the discovery of a new dark ring exterior to the bright rings of Saturn, by E. Schaer at the Observatory of Geneva.

In *A.N.* 4287, R. Jarry-Desloges shows that this ring was really discovered on 1907 September 5 and 7, at his observatory on Mount Revard, by Mr. G. Fournier.

This new ring is in part verified by observations made at Greenwich with the 28-inch refractor in October of 1908 (*M.N.*, lxix., November 1908), where a dusky ring was seen partly surrounding the bright rings. Though this was unmistakable in some of the observations, nothing is said as to the real width of the new ring, except that it was narrow. The Greenwich observations were made with powers of 450–550 and 670 on the 28-inch refractor. The object was apparently not difficult.

I observed Saturn carefully a number of times in 1908 as a continuation of my work of the year before (*M.N.*, lxviii. pp. 346–368). These observations did not show anything of the new ring, though it might have escaped notice at first through my not knowing of its existence. A careful search was made for it, however, on 1909 January 12 and 19, under good conditions, when nothing suggestive of its existence was seen. In November and the first part of December my whole attention was taken up with comet photography during the time that Saturn was in good position above the horizon, and no observations were obtained of the planet.

In all the observations of 1908 the rings of Saturn were carefully examined in hopes of finding something that would explain the presence of the luminous condensations seen on the dark part of the rings in 1907. When the conditions were good, therefore, the exterior dark ring ought to have been seen had it been as bright as when observed at Geneva and Greenwich.

As the observations of 1908 have a special bearing on some of the problems connected with the rings of Saturn, and especially with respect to this reported new dark ring, I will give them here, omitting those which, from poor conditions of seeing, are not important.

1908 July 19. Beautiful definition, magnifying power 700. The crape ring was unusually bright and the Cassini division well seen. The rings appeared flat and thin. The inner bright ring was very feebly brighter near the Cassini division. There was nothing to suggest the cause of the condensation seen last year. The shadow of the ring on the ball and the trace of the crape ring were well seen. The shadow of the ball was sharp and black. The south polar cap was quite dark. The ball was decidedly brighter than the rings.

Aug. 4. Definition good. The south polar cap was quite

dark and the crape ring very conspicuous. The outer ring at the ansæ appeared to be brighter than any part of the inner bright ring. There was nothing to suggest any cause for the condensations of last year. The only things at all suggestive were the crape ring and the Cassini division. The latter was not very well seen. Magnifying power 700.

Aug. 16. The southern cap was dark. The outer ring seemed to be brighter than the inner ring at the ansæ when best seen. The crape ring was quite bright.

Sept. 15. Seeing very good. The crape ring was quite easy. The south polar region was dark.

Sept. 20. Seeing very good. South polar region darkish. The crape ring quite bright. The rings looked very thin. At the ansæ the outer ring was the brighter of the two. Close south of the trace of the ring on the ball was a narrow line of shadow (shadow of rings). It was partly shadow and partly trace of the crape ring. The shadow was a thin edging to the south part of the trace of the rings, and lay close in between the inner edge of the bright rings and the edge of the crape ring. It must therefore have been seen wholly through the crape ring.

Oct. 6. Seeing fair. The crape ring was quite bright. Could see traces of the shadow of the rings on the ball towards the edges of the ball. This shadow was very thin. Could see nothing on the rings that would suggest the condensations of last year. Moisture on object-glass.

Dec. 27. Seeing good. Nothing abnormal.

1909 Jan. 12, $5^h\ 10^m$ to $5^h\ 40^m$. Looked carefully for the new outer dark ring of Saturn. Seeing = 3 (scale of 5). Eyepieces 460 and 700. Could see nothing of it. Specially examined at the ansæ. Occulted the planet by edge of field and by the smoked mica occulter. Nothing abnormal anywhere. The outer bright ring seemed to be a little brighter at the ansæ than the inner ring. There did not seem to be anything outside the bright rings—even close to the rings. A small satellite, south of the following ansa, was conspicuous. Observation quite satisfactory. The seeing was not perfect enough to give a sharply defined black image of the shadow of the rings on the ball.

Jan. 19, $5^h\ 30^m$ to $5^h\ 50^m$. Examined the planet carefully with 460 and 700. Seeing = 3. No evidence of a ring exterior to the bright rings. Then cut down aperture to 35 inches. Seeing = 3 - 4, much better. The shadow of the rings on the ball was sharp and black. Nothing abnormal about the ball or rings. The outer ring seemed a little brighter at the ansæ than the inner ring—perhaps due to contrast. Could not make much out of crape ring. It did not seem to be as bright as it was several months previous.

Conclusions.

One fact that impressed me in these observations, which were all made with the 40-inch telescope, was the unusual brightness of

the crape ring. It is precisely what should have occurred, however, for the ring was seen very obliquely, and a greater brightness would result from the fact that the same number of luminous particles were compressed by perspective into a much smaller space, and must therefore appear brighter. That this must be so was pointed out by Professor W. H. Wright in a paper in the *Astrophysical Journal*, vol. xxvii. p. 364.

Another fact was the change in relative brightness between the inner and the outer bright rings. When the rings are widely opened, the inner ring is decidedly brighter than the outer one. The present observations would seem to show that when the rings are seen at a low angle the outer ring appears the brighter of the two.

This peculiarity can also be explained by a similar assumption to that above used for the crape ring. That is, that there are fewer particles, area for area, in the outer than in the inner bright ring.

It seems to me that these observations alone could almost be taken as a proof of the meteoric constitution of the rings of Saturn.

It will be remembered that the light spots or condensations on the dark side of the rings (at their disappearance in 1907) were under observation here from 1907 July 2 for nearly five months before they were seen at any other observatory. No perfectly satisfactory explanation has yet been given for these luminous condensations. Two explanations were offered by me, *M.N.*, lxviii. pp. 355 and 365, the first of which in part seems improbable; the second connected them with the crape ring and the Cassini division. I am quite satisfied (and I think the observations in the present paper bear me out) that the inner condensations were due to the sunlight sifting through and being reflected from the particles of the crape ring. The outer condensations were in some manner connected with the Cassini division, as explained in the paper referred to above. The visibility of the dark side of the rings, it appears to me, was due to the sunlight passing through among the particles composing the rings, and being diffused and reflected from the dark side. I am now inclined to attribute most of this latter effect to the outer bright ring, in which the particles must be more thinly scattered. Indeed, I believe that if the surface of the rings was turned at right angles to us, the outer ring would be comparatively dark, and the crape ring would be perhaps invisible, from the apparent widening out of the spaces between the particles.

It is needless to say that the condensations were not due to thicker places on the rings. This was emphatically disproved by my observations both in July and October of 1907 and in January of 1908 when the rings were on edge to us. At these times the condensations should have been the most conspicuous if they were due to greater thickness. In reality, under these conditions, they entirely disappeared.

That they were due in some manner to the transmission of the sunlight through the rings (through the crape ring and Cassini

division, perhaps) is further shown by the fact that no trace of them was seen when the bright side of the rings was presented to us at a similar low angle to that of the dark side when they were visible.

Yerkes Observatory:
1909 *May* 3.

On the Erroneous Results of a Stereoscopic Combination of Photographs of a Comet. By E. E. Barnard.

The stereoscope enables one to see objects in perspective by the combination of two photographs taken from a slightly different point of view. One way to accomplish this effect is to shift the position and direction of the camera by a proper amount between the taking of the two pictures. Essentially the same effect would be produced by the bodily displacement of the object with the camera stationary, provided all of the background were itself too distant for parallactic displacement. Both these methods would require that there were no relative change in the individual parts of the object in the interval between the two pictures. In practice, stereoscopic pictures are made by simultaneous exposures with twin lenses separated by a few inches. For celestial bodies, where our base line is too small to give any appreciable parallax, the last of the two methods is resorted to. In the case of the Moon the perspective is obtained by the aid of libration; and as the phase has to be exactly the same, a very long interval is required. In the case of a bright asteroid an interval of an hour, or of a proper motion star of several years, will produce the required effect. The short interval is also applicable to a comet, and beautiful and startling effects are produced by this means in the case of a comet with a tail. Few bright comets, however, are above the horizon long enough to permit the two photographs to be made for this purpose. On account of its high north declination, and its consequent visibility through all or nearly all the night, comet *c* 1908 (Morehouse) was specially suited for stereoscopic photographs, and the material acquired for this purpose is abundant. The combinations of proper sets of these pictures show the comet in beautiful relief suspended alone in space, as we know it is in reality, with the various parts of the tail in individual perspective. There is a wonderful effect of reality in these pictures, and the filmy, breath-like character of the comet is shown as no single picture can even hope to show it. These photographs are marvellously instructive, and one is tempted to deduce certain "facts" from the appearance which are probably entirely erroneous.

This is such a serious matter, and one so fascinatingly delusive, that I have thought it worth while to draw attention to the chances for serious errors that are inherent in the very success of the experiment in the case of a comet.

I have combined a number of photographs of the present comet

in which the motion in an hour or so, between the exposures, permits stereoscopic effect. One of the most remarkable of these combinations is that of 1908 October 15. On this date there was a sudden twist or break in the tail, which formed irregular cloud-like masses that moved out from the comet along the general direction of the tail. In the stereoscope these two pictures produce an exquisite object suspended in front of the stars. Apparently it is easy to see which are the farther and which the nearer parts of the comet. The south end of the long irregular mass at an angle to the tail is the nearer. From this the tail has a twisted or corkscrew form, the nearer and farther convolutions of which are at once evident. The straight stem-like tail emanating from the head passes behind this mass. On each side of the head is a small stream shot out at a considerable angle to the main tail. The south one of these apparently recedes from us at a large angle to the plane of the photograph. There are other peculiarities which will be apparent to anyone viewing the picture, and which cannot be well described here.

But how much of this perspective is real? I believe that there is but little of it that can be true. In the first place, these masses were receding from the comet and changing their actual forms, and especially their position angles, so that a pseudo-stereoscopic effect would be produced, and what is really the nearer portion of the comet may appear to be a distant part. Some of the features where the details were not changing rapidly—as, for instance, the short tail from the south side of the head—would probably not be affected much by the outgoing particles, and a true stereoscopic effect would be produced. I have my doubts about much of the rest. It would be an interesting problem to differentiate the effect of the individual motion of the masses from the parallactic effect, and thus be able to tell which are the nearer and which the distant parts of the comet.

In preparing this stereogram (October 15) much trouble was experienced in making the two pictures combine. The outward motion of the masses and their angular motion separate them too much for the best result. By reducing the scale of the images considerably this difficulty was overcome and a fairly good combination obtained. This trouble did not occur with the plate of November 16, because the details were of a different nature and easily reconcilable.

All of the pictures were made with the 10-inch Bruce lens. The times and durations for these are—

		G.M.T. h m	Duration. h m
1908 October 15	No. 1	12 58	1 22
	No. 2	14 31	1 30
1908 November 16	No. 1	12 14	1 5
	No. 2	13 46	1 50

In connection with this subject I would say that I never need the stereoscope to combine these pictures. By a certain control of my eyes, any stereoscopic picture is far more beautiful without the stereoscope, and I never use the instrument. The magnification of the image is somewhat less. Indeed I have no trouble in perfectly combining thus the pictures of October 15, even when they are somewhat enlarged from the original. When one has this command over his eyes he will never again use a stereoscope.

In the stereogram of November 16 the two narrow streams emanating from the head on the left-hand (south) side are apparently ejected from the comet in a direction some 20° or 30° from the plane through the head, parallel to the plate, but away from us and beyond that plane. Those on the right or north seem to be ejected toward us, but this is rather uncertain. The entire comet has a beautiful filmy appearance, and some of the structures seem to be due to broad sheets of matter. I have compared Nos. 1 and 3 of this date, but the changes have been too large to secure good adjustment.

The comparisons of November 18 do not produce such a striking and beautiful effect, though it is very good.

While I believe that a good deal of the perspective on these stereograms is untrue, and that it does not accurately represent the actual appearance of the comet at the time, it must not be understood that I would discourage the stereographic study of comets. I believe that much can be learned of the general structure of these bodies from such a study. We must simply be guarded as to the relative perspective of the various parts of the comet at that time, as resulting from a combination of the two photographs.

Though the appearance, in a stereograph, of any one comet may be partly false, there is certainly no other method that can show us how a comet really looks in space; and for this reason, if for no other, it will, I believe, in a truthful manner, help us to understand the features of comets in general.

Yerkes Observatory, Williams Bay, Wisconsin:
1909 *May* 25.

Note by the Secretaries.

Professor Barnard has most kindly presented to the Society three stereoscopic combinations of photographs to illustrate this paper. As much of the delicacy of structure would necessarily be lost in any printed reproduction, it has been decided to make photographic copies for viewing in a stereoscope, and place them on sale to Fellows.

Prices: Glass transparencies, . 6s. set of three.
Prints mounted on cards, 3s. ,,

Note on an Electric Heater for use in a plate-holder on damp nights.

(*Communicated by the Astronomer Royal.*)

This electric heater was devised by Mr. Davidson and Mr. Melotte in 1908 November for the purpose of keeping the photographic plate dry during a prolonged exposure with the 30-inch reflector.

Whilst Comet *c* 1908 was under observation it was noticed that when the photographic plate was taken from the telescope at the end of an exposure it seemed to be quite damp. The air at the time was very humid, and the gelatine of the plate (which is quite exposed at the mouth of the tube of the 30-inch reflector) had evidently absorbed a good deal of moisture from the air. The effect on the plate was an appreciable diminution in sensitiveness, shown by the fading of the star-trails as the exposure proceeded.

The electric heater was designed to remove this difficulty, as it is evident that on damp nights a limit was set to the effective exposure.

It consists of a grid of resistance wire stretched on a vulcanite frame nearly in contact with an aluminium plate, and placed in the back of the dark slide, the metal plate in contact with the sensitive plate. In the case of the plate-holder for 16 × 16 cm. plates, about 7½ yards of No. 38 "Platinoid" wire is used for the grid, giving a resistance of about 130 ohms. The current used is about 16 ampères obtained from the lighting circuit at 100 volts, a 5 c.p. lamp being introduced in series as a resistance. This amount of current is sufficient to make the plate just perceptibly warm to the touch, and experience has shown this to be enough to keep the plate dry, while at the same time the definition does not suffer from the heated air, as might happen if the plate were made unduly hot.

Royal Observatory, Greenwich:
1909 *June* 8.

Report on the Measurement of an Arc of Meridian in Uganda.

By Col. C. F. Close, R.E.

[At their meeting on 1907 April 12 the Council voted the sum of £50 as a small contribution from the funds of the Society towards the cost of measuring a section of the great African arc of meridian on the Uganda-Congo boundary. The Council have now much pleasure in publishing to the Society the annexed short preliminary report from Col. C. F. Close, announcing the successful completion of the work.]

1. The measurement was commenced in March 1908 and completed in February 1909. The arc extends from 1° 10′ N. to 1° 10′ S., *i.e.* the length is $2\frac{1}{3}$ degrees, or about 165 miles.

2. One base, length 11 miles, was measured in the northern portion, in the Semliki Valley. The chain consists of 1 complex figure, 3 quadrilaterals, and 1 tetragon.

3. All the stations have been marked in a permanent manner, and the Government of Uganda has been notified of their positions.

4. The probable error of an observed angle is about 0″·4.

5. Three azimuths and 14 latitudes were observed.

6. Magnetic observations for declination and dip were made at twenty stations.

7. The work was organised by Major Bright, C.M.G., and carried out by a British party consisting of Captain Jack, R.E., Mr. McCaw, Mr. C. Chevallier, L.-Cpl. Jones, R.E., L.-Cpl. Page, R.E.; and for a portion of the time, Captain S. Iredell, 4th Battalion, King's African Rifles, who also commanded the escort.

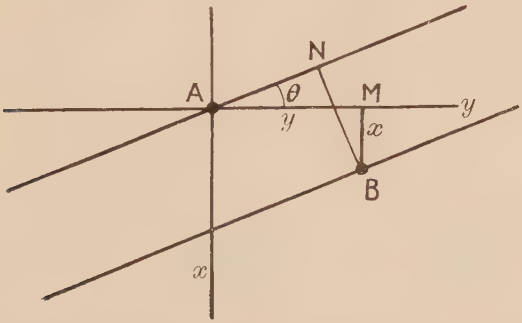
The Belgian party consisted of Captain Wangermée and Dr. M. Dehalu.

1909 *June* 1.

A Method of Double-Star Measurement. By J. B. Dale, M.A.

1. The method of double-star measurement in general use consists of a direct determination of the polar coordinates of one of the stars relative to the other. The operations involved are, theoretically at any rate, very simple, but the comparison of the same observer's observations made on different occasions, or of the observations of different observers, reveals the existence of well-defined systematic errors both in distance and position angle. The cause of these errors is somewhat obscure, and their elimination is difficult and uncertain.

It is, however, a well-established principle that the best method of eliminating systematic error of any sort is to vary as far as possible the conditions of measurement, and the object of the following note is to suggest a method of measurement in which such variation is made. It is possible that results obtained in this way may serve as a check upon the results obtained by the ordinary method of procedure, and may throw some light on the source and character of the errors involved.



2. In the figure, A and B represent two stars whose relative position is to be determined. Take A as an origin of rectangular coordinates, the positive direction of the x axis being drawn towards the north and the y axis being the parallel. Then (x, y) being the coordinates of B, the distance of B from a line drawn through A making a positive angle θ with Ay is

$$x \cos\theta + y \sin\theta = \mathrm{BN}.$$

This distance may be measured in the usual way with the micrometer, one wire A′ being set on A and the other B′ on B.

If a is the reading of B′ when it coincides with A′, then the reading when on B is

$$z = x \cos\theta + y \sin\theta + a.$$

Theoretically, the three unknowns x, y, a can be found from measurements made with three different values of θ, but of course it is preferable to increase the number of settings and deduce the

values by a least square solution of the resulting equations. I have found it convenient to assign to θ the values 0°, 30°, 60° . . . . 330°, and so obtain twelve values of z, z_1, z_2 . . . . z_{12}. The values of x and y may then be found in the usual manner.

3. The labour of solution is much simplified by arranging the work as follows:—

	z_1	z_2	z_3	z_4	z_5	z_6
	z_7	z_8	z_9	z_{10}	z_{11}	z_{12}
Sums	s_1	s_2	s_3	s_4	s_5	s_6
Differences	d_1	d_2	d_3	d_4	d_5	d_6

	d_1	d_2	d_3	d_4
		d_6	d_5	
Sums		d_2+d_6	d_3+d_5	
Differences		d_2-d_6	d_3-d_5	

Then
$$x=\tfrac{1}{6}[d_1+\sqrt{\tfrac{3}{2}}(d_2-d_6)+\tfrac{1}{2}(d_3-d_5)]$$
$$y=\tfrac{1}{6}[d_4+\sqrt{\tfrac{3}{2}}(d_3+d_5)+\tfrac{1}{2}(d_2+d_6)]$$
$$a=\tfrac{1}{12}\Sigma(s)$$
also
$$\rho=\sqrt{x^2+y^2} \quad \text{and} \quad \phi=\tan^{-1}(y/x).$$

In a perfect set of measurements the s's would be identical. Further, we should find

$$\begin{aligned} d_1 &: \sqrt{\tfrac{3}{2}}(d_2-d_6) &&: \tfrac{1}{2}(d_3-d_5) \\ =d_4 &: \sqrt{\tfrac{3}{2}}(d_3+d_5) &&: \tfrac{1}{2}(d_2+d_6) \\ =2 &: 3 &&: 1 \end{aligned}$$

If the residuals of the z's be formed, and ϵ be the mean error of the z's, the mean error of x, y or ρ is $\epsilon/\sqrt{6}$ and of a, $\epsilon/\sqrt{12}$. The mean error in ϕ is $\epsilon/(r\sqrt{6})$ radians or $(23{\cdot}4\ \epsilon/r)°$.

4. The following set of measures of γ Leonis made on May 23, 1909 will illustrate the procedure. The unit is one division of the micrometer screw head, and is equal to $0''{\cdot}4208$. The telescope has an aperture of $3\frac{1}{2}$ inches.

		Sums.	Diffs.
$z_1=50{\cdot}4$	$z_7=58{\cdot}9$	109·3	− 8·5
$z_2=54{\cdot}3$	$z_8=53{\cdot}8$	108·1	+ 0·5
$z_3=59{\cdot}9$	$z_9=49{\cdot}6$	109·5	+10·3
$z_4=62{\cdot}7$	$z_{10}=46{\cdot}2$	108·9	+16·5
$z_5=64{\cdot}0$	$z_{11}=46{\cdot}0$	110·0	+18·0
$z_6=61{\cdot}8$	$z_{12}=47{\cdot}2$	109·0	+14·6

$d_1=-8{\cdot}5$	$d_4=16{\cdot}5$
$\sqrt{\tfrac{3}{2}}(d_2-d_6)=-13{\cdot}2$	$\sqrt{\tfrac{3}{2}}(d_3+d_5)=24{\cdot}6$
$\tfrac{1}{2}(d_3-d_5)=-3{\cdot}9$	$\tfrac{1}{2}(d_2+d_6)=7{\cdot}6$
$x=-4{\cdot}27$	$y=+8{\cdot}12$

whence $\rho=9{\cdot}174=3''{\cdot}86$, $\phi=117°{\cdot}8$.

The residuals $z_o - z_c$ are as follows:—

+·1	−·6	+·4	0	+·2	−·5
+·1	−·4	−·1	−·3	+·6	+·4

The mean error in z is 0·32 = 0″·135.
The mean error in x, y, ρ is 0·13 = 0″·055.
The mean error in ϕ is . . 0°·82.
The correction for the parallel was negligible.

5. It is not yet possible to state any definite results obtained by this method, but measures of the same stars made on different nights suggest the following:—

(1) The mean error in distance derived from one complete set of measures is fairly constant, the average being 0·25 of a division.

(2) The mean discordance between measures made on different nights is somewhat larger; it averages 0·30 of a division, and appears to increase with the distance.

(3) The mean error of the derived values of the position angle derived from one night's measures is about twice as large as the mean discordance between measures on different nights.

(4) The residuals frequently show a periodic run, those between 0° and 180° being repeated between 180° and 360°. They could be well represented by terms of the form

$$a \cos 2\theta + b \sin 2\theta.$$

These results are in agreement with the general experience of observers, but it is rather remarkable that measures on different nights lead to much smaller alterations in the ratios of the z's than the differences in their absolute values would lead one to expect. There seems to be an alteration in scale value. The discordances are larger than slight variations in adjustment for focus could produce, but the periodic run in the residuals leads one to think that it may be a parallactic effect, due to the stars and wires not being together in focus.

King's College, London:
1909 *June* 7.

Magnitude of η Argus, 1909. By R. T. A. Innes.

The comparison stars were, as in former years :—

C. G. A. Cluster Catalogue No. 121,	8·00 mag.	8 colour.
Gilliss 1332, ...	7·60 ,,	4 ,,

The separate observations are :—

1909 Mar. 31	7·75 mag.	5 colour.
Apl. 11	7·8 ,,	6 ,,
,, 22	7·8 ,,	6 ,,

A low power on the Grubb 9-in. refractor was used. The means of the observations by years, which I have made since 1896, are as follows :—

	Mag.	Colour.	No. of Obs.
1896·4	7·58	5·0	4 – 1
1899·5	7·71	7·0	4 – 2
1900·3	7·68	6·3	3
1901·3	7·78	6·8	3
1902·1	7·72	6·5	3
1905·5	7·67	7·5	5 – 2
1908·4	7·75	5·7	4 – 3
1909·3	7·78	5·7	3

These observations show no variation since 1899 greater than the errors of estimation.

Government Observatory, Johannesburg :
1909 *April* 24.

Note on certain Coefficients appearing in the Algebraical Development of the Perturbative Function. By R. T. A. Innes.

(*Originally written in January* 1909 (*but lost in post*), *and rewritten* 18*th. March* 1909.)

I.

We find a short algebraical development of the Perturbative Function in terms of the mean anomalies in Laplace's *Mécanique Céleste*. This was extended by Burckhardt and de Pontécoulant. Later, Le Verrier gave a development to the 7th. powers of the eccentricities and 6th. powers of the inclinations (both inclusive), and this was extended to the 8th. powers by Bouquet. Le Verrier gave extended tables and numerous precepts with his development (*Les Annales*, i. and x.), and in consequence it has been widely used.

A more methodical and general development was given by Newcomb (*Ast. Papers*, v., parts 1 and 4, 1895). Earlier, Newcomb had published a development in terms of the eccentric anomalies (*Ast. Papers*, iii.) which he used in deriving the periodic perturbations of the four inner planets, but it would seem that the further adoption of this development has been tacitly condemned by its illustrious author—it appears to be necessary to state this explicitly, as in a recent didactive treatise on this subject (I refer to Mons. H. Poincaré's *Leçons de Méc. Cél.*, ii., part 1, 1907) the author's references to Newcomb omit his later and more generally useful papers.

The earlier developments (including Le Verrier's) make use of the Laplace coefficients

$$b_{\frac{1}{2}}^{(i)},\ b_{\frac{3}{2}}^{(i)},\ b_{\frac{5}{2}}^{(i)},\ \text{etc. etc.},$$

and of their derivatives of the first and higher orders. An inspection of Le Verrier's formulæ will show that the coefficients which actually appear are in a slightly different form, viz.

$$b^{(i)},\ \alpha b_{\frac{3}{2}}^{(i)},\ \alpha^2 b_{\frac{5}{2}}^{(i)},\ \text{etc. etc.},$$

and that the derivatives of these are there implicitly.

Finding the numerical values of the various Laplace coefficients (especially if α is over $\frac{1}{2}$ and accuracy is required) is a wearisome process, but when α is less than $0{\cdot}75$ the actual work required can be lessened by using tables by Runkle and others. Well-known recurrence formulæ exist between the various coefficients, but unfortunately their use leads to accumulating errors.* Le Verrier

* This was pointed out by Le Verrier in one of his earliest papers (Paris, 1841) and is well known, yet we find Mons. H. Poincaré, in the work already cited, recommending their use. Of course the recurrence formulæ are *mathematically* exact, but they are too inaccurate for astronomical use, paradoxical as it sounds!

introduced transformations of power-series so as to facilitate their calculation, and gave extensive tables (*Les Annales*, ii.), but even Le Verrier's figures will not give the 5th. place correctly if a is so large as 0·825 (Titan-Hyperion).

Besides the recurrence formulæ, Le Verrier gave (*Les Annales*, ii. pp. 108–9) several isolated relations between groups of the coefficients, such as

$$\left(4a\frac{d}{da}+14a^2\frac{d^2}{da^2}+8a^3\frac{d^3}{da^3}+a^4\frac{d^4}{da^4}\right)b_{\frac{1}{2}}^{(0)}=\left(2a\frac{d}{da}+4a^2\frac{d^2}{da^2}+a^3\frac{d^3}{da^3}\right)b_{\frac{3}{2}}^{(1)}.$$

He found ten such relations, but gave no indication that a generalisation exists. The advantage of such relations is that they consist of simple additions of tabulated quantities multiplied by integers. The recurrence formulæ previously mentioned involve awkward multiplications by fractional numbers and by $\frac{1}{a}+a$, $1+na^2$, etc.

II.

The object of this paper is to give a general formula which leads to a very simple recurrence formula, and embraces the generalisation mentioned above. This will be done for Newcomb's development. I will use the symbol introduced recently by Cowell—it has the great advantage of avoiding the use of an affix in the usual place of a power-index. The coefficients will compare as follows :—

Laplace.	Newcomb.	This paper.
$b_{\frac{1}{2}}^{(i)}$	$c_1^{(i)}$	$b_{i,1}$
$a\,b_{\frac{3}{2}}^{(i)}$	$c_3^{(i)}$	$b_{i,3}$
$a^2\,b_{\frac{5}{2}}^{(i)}$	$c_5^{(i)}$	$b_{i,5}$
etc.	etc.	etc.

In place of the ordinary derivatives, Newcomb has introduced logarithmic derivatives, indicated by the operator $\mathrm{D}=a\frac{d}{da}$. The general relation is

$$\left(a\frac{d}{da}\right)^n=\mathrm{D}(\mathrm{D}-1)(\mathrm{D}-2)\ \ldots\ (\mathrm{D}-n+1),$$

and it enables us to pass from one set of derivatives to the other if wanted.

The recurrence formula used by Newcomb

$$c_{n+2}^{(i,j)}=\frac{n+2j+na^2}{n^2a(1-a^2)}\,c_n^{(i,j+1)}+\frac{(n+2j)(2i+2-n)}{(n^2-a^2)}\,c_n^{(i+1,j)}$$

is free from some of the defects of that of Laplace. The quantity

$c_n^{(i,j)}$ is an intermediary between $D^j c_n^{(i)}$ and $c_n^{(i)}$, which was introduced by Newcomb to facilitate calculation.* It is, to a simple factor, a hypergeometric series. If we apply the Gaussian relation

$$F_{\beta+} - F = \frac{\alpha}{\gamma}\, x F'''$$

to this quantity we get

$$(n-2)^2\, c_n^{(i+1,j)} = (2n+j-1)c_{n-2}^{(i,j+1)} + c_{n-2}^{(i,j+2)},$$

a recurrence formula freed from all defects. But what we really require are the quantities $c^{(i)}$, $Dc^{(i)}$, $D^2c^{(i)}$, etc. If we apply the formulæ given by Newcomb to find these quantities (*Ast. Papers*, v. p. 313, etc.) the last recurrence formula becomes

$$(n-2)^2 D^m c_n^{(i)} = D^m\left(D + i + \frac{n-3}{2}\right)\left(D - i - \frac{n-5}{2}\right)c_{n-2}^{(i-1)},$$

or using the newer notation

$$(n-2)^2 D^m b_{i,n} = D^m\left(D + i + \frac{n-3}{2}\right)\left(D - i - \frac{n-5}{2}\right)b_{i-1,n-2}.$$

Particular cases are—

$$\begin{aligned}
&D^m c_3^{(0)} = D^m(D^2 + D - 0)c_1^{(1)} \\
&D^m c_3^{(1)} = D^m(D^2 + D - 0)c_1^{(0)} = D^m(D^2 + D - 2)c_1^{(2)} \\
&D^m c_3^{(2)} = D^m(D^2 + D - 2)c_1^{(1)} = D^m(D^2 + D - 6)c_1^{(3)} \\
&\quad \ldots \qquad \ldots \qquad \ldots \\
&9D^m c_5^{(0)} = D^m(D^2 + D - 0)c_3^{(1)} \\
&9D^m c_5^{(1)} = D^m(D^2 + D - 2)c_3^{(0)} = D^m(D^2 + D - 0)c_3^{(2)} \\
&\quad \ldots \qquad \ldots \qquad \ldots \\
&25D^m c_7^{(0)} = D^m(D^2 + D - 2)c_5^{(1)} \\
&25D^m c_7^{(1)} = D^m(D^2 + D - 6)c_5^{(0)} = D^m(D^2 + D - 0)c_5^{(2)} \\
&\quad \ldots \qquad \ldots \qquad \ldots \\
&49D^m c_9^{(0)} = D^m(D^2 + D - 6)c_7^{(1)} \qquad \text{etc. etc.}
\end{aligned}$$

With these new formulæ we may check some of the figures furnished by Newcomb (*Ast. Papers*, v. p. 343) for Venus-Earth. We have

	$D^4 c_3^{(3)} =$ 66839	(Newcomb)
Check	$D^6 c_1^{(2)} =$ 64691	
	$D^5 c_1^{(2)} =$ 4182	
	$-6D^4 c_1^{(2)} =$ −2033·4	
	Hence 66839·6 $= D^4 c_3^{(3)}$.	

* It was also used in the same sense by W. Scheibner (*Störungsfunction*, Gotha, 1853)

If $D^4c_3^{(4)}$ is thus calculated, it is found $=65583$, whereas Newcomb has 65850, but an independent calculation has refound the former figure.

Again,

$$Dc_9^{(4)} = \quad 60000 \qquad \text{Newcomb (apparently the nearest round figure which is sufficiently accurate).}$$

Check

$$D^3c_7^{(3)} = \ 2974000$$
$$D^2 \text{,,} = \ 118490$$
$$-42D \text{ ,,} = -224952$$
$$1/49 \times 2867538 = 58521 = Dc_9^{(4)}.$$

From the general formula many simple relations may be deduced, such as

$$D^m(b_{0,3} - b_{2,3}) = 2D^m b_{1,1}$$
$$3D^m(b_{0,5} - b_{2,5}) = 2D^m b_{1,3}$$
$$5D^m(b_{0,7} - b_{2,7}) = 2D^m b_{1,5} \qquad \text{etc. etc.}$$

The importance of the general formula in the development of the Perturbative Function is considerable. It not only simplifies the calculation of $b_{i,3}$, $b_{i,5}$, etc., but it can also be turned to use in computing the $D^m b_{i,1}$ coefficients, which will save the introduction of the $c^{(i,j)}$ quantities.

We have Newcomb (*Ast. Papers*, v. p. 310), and noting that

$$Dc_1^{(i)} = ic_1^{(i)} + c_1^{(i,1)}$$

$$\tfrac{1}{2}(D - i)b_{i,1} = \frac{1.3.5 \ . \ . \ . \ 2i+1}{2.4.6 \ . \ . \ . \ 2i+2}\alpha^{i+2}F(\tfrac{3}{2}, \tfrac{3}{2}+i, 2+i, \alpha^2)$$

which transforms into

$$\tfrac{1}{2}(D - i)b_{i,1} = \frac{1.3.5 \ . \ . \ . \ 2i+1}{2.4.6 \ . \ . \ . \ 2i+2}\alpha^i\beta^2F(\tfrac{1}{2}, \tfrac{1}{2}+i, 2+i, \alpha^2)$$

wherein

$$\beta^2 = \frac{\alpha^2}{1 - \alpha^2}.$$

Tisserand (*Méc. Cél.*, i. p. 274) shows that

$$b_{i,1} = \alpha^i\frac{2}{\pi}\int_0^\pi \frac{\sin^{2i}\theta \, d\theta}{\sqrt{1 - \alpha^2\sin^2\theta}}$$

and we also have

$$F(\tfrac{1}{2}, \tfrac{1}{2}+i, 2+i, \alpha^2) = N\frac{2}{\pi}\int_0^\pi \frac{\sin^{2i}\theta - \sin^{2i+2}\theta}{\sqrt{1 - \alpha^2\sin^2\theta}}d\theta,$$

wherein N is the ratio of two whole numbers.

Hence

$$(D - i)b_{i,1} = (2i+1)\alpha^i\beta^2\left(\frac{b_{i,1}}{\alpha^i} - \frac{b_{i+1,1}}{\alpha^{i+1}}\right),$$

and $Db_{i,1}$ can be found from $b_{i,1}$ and $b_{i+1,1}$.

Later, it will be seen that we should compute

$$p_i = \frac{b_{i,1}}{\alpha^i},$$

and this must be done with precision to avoid the introduction of error.

To find the higher derivatives we can use the differential equation satisfied by the coefficients (Tisserand, *Méc. Cél.*, i. p. 281) which in the new notation is as follows—

$$(D^2 - i^2)b_{i,1} = \beta^2(2D + 1)b_{i,1},$$

and noting that

$$D\beta^2 = 2\beta^2(1 + \beta^2)$$

we have

$$D(D^2 - i^2)b_{i,1} = \beta^2[4D^2 + 5D + (2 - 2i^2)]b_{i,1}$$
$$D^2(D^2 - i^2)b_{i,1} = \beta^2[6D^3 + 13D^2 + (12 - 4i^2)D + (4 - 4i^2)]b_{i,1},$$
$$\text{etc.} \qquad \text{etc.}$$

To obtain either $D^{m+2}b_{0,1}$ or $D^{m+2}b_{1,1}$ we must use these formulæ; for other values of i there is a better process. Between $D^m b_{0,1}$ and $D^m b_{1,1}$ we have the following simple relations—

$$Db_{0,1} = \alpha D b_{1,1}$$
$$D^2 b_{0,1} = \alpha D(D + 1)b_{1,1}$$
$$\cdots \qquad \cdots$$
$$D^m b_{0,1} = \alpha D(D + 1)^{m-1} b_{1,1}.$$

To obtain $D^{m+2}b_{i,1}$ for other values of i we can write the formula—

$$D^m b_{i,n} = D^m\left[D^2 + D - \left(i - \frac{n-5}{2}\right)\left(i + \frac{n-3}{2}\right)\right]b_{i-1,n-2}$$

first with i positive and then with i negative. Equating the two values so found and confining our attention to the case of $n = 1$ yields

$$D^m(D^2 + D - 2)b_{2,1} = D^m(D^2 + D - 0)b_{0,1}$$
$$D^m(D^2 + D - 6)b_{3,1} = D^m(D^2 + D - 2)b_{1,1}$$
$$\text{etc.} \qquad \text{etc.}$$

so that in actual work we have

$$D^{m+2}b_{2,1} = D^m(D^2 + D - 0)b_{0,1} - D^m(D - 2)b_{2,1}$$
$$D^{m+2}b_{3,1} = D^m(D^2 + D - 2)b_{1,1} - D^m(D - 6)b_{3,1}$$
$$D^{m+2}b_{4,1} = D^m(D^2 + D - 6)b_{2,1} - D^m(D - 12)b_{4,1}$$
$$\text{etc.} \qquad \text{etc.}$$

Thus all the quantities except the $b_{i,1}$ can be derived quite simply once the latter are found, no large numbers nor awkward factors are involved, and there is no accumulation of error except that which is inevitable in any extended calculation, and which arises from the uncertainty of the last decimal place.

The formulæ developed above are useful in computing the action of the planets on the Moon, and evade a difficulty to which Brown has called attention (*Action of Planets*, p. 34).

It will be apparent that the introduction by Newcomb of the operator D instead of $\alpha\frac{d}{d\alpha}$ is not only an analytical improvement, but one of great practical importance as well.

I am now finishing a numerical application to the case of Titan-Hyperion, for which α exceeds 0·825, in which the whole process, with checks and alternative methods, is fully illustrated.

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Numerical Example. (Innes.)

Calculation of $b_{0,1}$.

This is the most important constant amongst the coefficients $D^m b_{i,n}$, as its value enters into all of them.

Earlier investigators computed its value by means of the hypergeometrical series $F(\frac{1}{2}, \frac{1}{2}, 1, \alpha^2)$, which is not very converging, especially for $\alpha > 0{\cdot}707$. Le Verrier gave a transformation of this series (*Annales*, ii. p. [5]), which was used by Runkle in his well-known tables, but the transformation is still unsuited for a rapid and exact calculation, and in fact Runkle's 7th. decimal place is frequently in error; thus for $\alpha = 0{\cdot}75$ Runkle gives

$$\log b_{01} = 0{\cdot}3861679,$$

whereas the true value is $= 0{\cdot}3861685$.

Newcomb, in his planetary theories (*Astr. Papers*, iii. p. 69), recommends a formula suggested by the use of Gauss's arithmetico-geometrical mean, viz.

$$\log b_{01} = N - \log\cos\tfrac{1}{2}\theta - \tfrac{1}{4}\log\cos\theta \qquad (\alpha = \sin\theta)$$

wherein N is very nearly equal to $\log 2$. A small table of N is given by Newcomb, but it has evidently been computed by means of Runkle's table, as it suffers from the same errors. The value of N is probably equal to

$$\log 2 + 2\log(1 - 2q^4 + 2q^{16} - \text{etc.})$$

wherein q is the quantity known as Jacobi's Nome.

I have preferred to calculate the value of $b_{0,1}$ by means of the following formulæ, which have been adapted from Verhulst (*Fonctions Elliptiques*, Bruxelles, 1841, p. 256):—

$$b_{0,1} = \frac{8}{(1+\sqrt{\alpha_1})^2}(1 + 2q^4 + 2q^{16} + \text{ etc.})^2$$

or

$$= \frac{8}{\left(\frac{1+\sqrt{\alpha_1}}{2} + \sqrt[4]{\frac{(1+\alpha_1)\sqrt{\alpha_1}}{2}}\right)^2}(1 + 2q^{16} + 2q^{64} + \text{ etc.})^2.$$

It seems inadvisable to replace α_1 by $\cos\theta$.

In the first of these two formulæ, the terms in q amount to less than one unit in the 8th. place of decimals if α does not exceed 0·35; in the second formula the terms in q will only amount to one unit in the 8th. place of decimals if α is as large as 0·998, a value much larger than is ever met with. In the case of Titan-Hyperion ($\alpha = 0{\cdot}825$), the use of the second formula, *neglecting* the q terms, introduces an error of 7 in the 19th. decimal place of the logarithm of $b_{0,1}$.

If we write

$$\log b_{0,1} = \log 8 - 2\log(1+\sqrt{\alpha_1}) + N_1$$

the quantity N_1 may be taken from the table on p. 649.

The following values of $\log q$ for different values of α will afford a rough estimate of the q terms in $b_{0,1}$:—

$\alpha = 0{\cdot}05$	$\log q = 6{\cdot}20$	$\alpha = 0{\cdot}35$	$\log q = 7{\cdot}91$	$\alpha = 0{\cdot}65$	$\log q = 8{\cdot}53$
0·15	7·15	0·45	8·15	0·75	8·71
0·25	7·61	0·55	8·35	0·85	8·90

(*N.B.*—10 has been added to $\log q$.)

As a numerical example we compute $\log b_{0,1}$ for $\alpha = 0{\cdot}75$ by the approximate formula

$$\log b_{0,1} = \log 8 - 2\log\left[\frac{1+\sqrt{\alpha_1}}{2} + \sqrt[4]{\frac{(1+\alpha_1)\sqrt{\alpha_1}}{2}}\right].$$

$$\alpha = \frac{3}{4} \quad . \quad \alpha_1 = \frac{\sqrt{7}}{4} \quad . \quad . \quad . \quad . \quad \log\alpha_1 = 9{\cdot}82048903$$

$$\frac{1+\sqrt{\alpha_1}}{2} = 0{\cdot}90664414 \qquad \sqrt[4]{\frac{(1+\alpha_1)\sqrt{\alpha_1}}{2}} = 0{\cdot}90661866$$

$$\log 8 = 0{\cdot}90308999$$
$$2\log 1{\cdot}81326280 = 0{\cdot}51692150$$
$$\log b_{0,1} = 0{\cdot}38616849$$

It will be noticed that although q has been introduced into the formulæ, it is in reality not used (owing to the extraordinarily close

approximation to the true value given by the second formula); thus it does not seem necessary to repeat the equations by which q may be found from a; they will be found on p. 494 of the *M.N.*, vol. lxii., where, however, k_1 is written for a_1.

The next step is to calculate $b_{i,1}$ from $i=1$ to such a value of i as experience may indicate. Having found $b_{0,1}$, there is no better way of finding the other values than that first shown by Hansen, and more fully and simply set out by Newcomb (*Ast. Papers*, iii. p. 64). The process here followed only deviates in unessentials from that of Newcomb. I prefer to write

$$\frac{b_{i,1}}{a^i} = b_{0,1}\, p_1\, p_2\, p_3 \ldots p_i,$$

because the quantities p change slowly for considerable changes in a. At first sight the computation of these quantities appears complicated, but on closer acquaintance it is found most simple. Once it is started, the only table required is Zech's subtraction-table, and in general only one opening of that. The one disadvantage is that Zech's table has at most only 7 decimal places, and no more extended table is available; hence it is not possible to be certain of the last figure; in spite of this, the exactitude is quite sufficient in practice.

The quantities $b_{0,1}$ and $p_1, p_2, \ldots p_i$ are computed so quickly and easily that tables giving them with a as argument seem to be superfluous. To exhibit, however, the march of these quantities, Mr. Frank Robbins has computed the short table on p. 650. Short as this table is, it will permit the p_i to be taken out almost correctly by inspection.

In the case of Titan-Hyperion, for which $a=0{\cdot}8250634$ (H. Struve), the following are the chief figures involved—

$$\log a = 9{\cdot}91648732$$

$1-a^2=a_1^2=0{\cdot}31927039$	$\log = 9{\cdot}50415864$
$1+\sqrt{a_1}\quad =1{\cdot}75169154$	$\log = 0{\cdot}24345763$

Hence by the formulæ on p. 639,

$$b_{0,1} = 2{\cdot}6074655 \qquad \log = 0{\cdot}41621857$$

To calculate p_{11} use is made of the well-known Gaussian continued-fraction. In this calculation the letters a, b, c, d . . . . and F are simple fractional factors; Zech is put against the figures for which Zech's subtraction-table is used. The calculation is quite similar to the example given by Newcomb (*Ast. Papers*, iii. p. 71), but is more extended because of the large value of a.

co-log		$-$ 0·34	$-$ 1·486	$-$ 0·2803	$-$ 1·75292	$-$ 0·231014	$-$ 2·5051152
Zech		0·265	0·0144	0·32199	0·007740	0·3845441	0·0013594
f	9·83	e 8·416	d 9·8716	c 8·09211	b 9·928271	a 7·2773661	F 9·9797966
a^2	9·83	9·833	9·8330	9·83297	9·832975	9·8329746	
	9·66	$-$ 8·514	$-$ 9·7197	$-$ 8·24708	$-$ 9·768986	$-$ 7·4948848	$-$ 9·9811560 $=p_{11}$

With this value of p_{11} we proceed

p_{11}	=	9·9811560
$a^2/(1+a^2)$		9·6074768
Constant		0·0211893
		9·6098221
co-log		0·3901779
Zech		0·2271017
Constant		9·9777236
$1/(1+a^2)$		9·7745021
p_{10}		9·9793274

and so on; the only tabular figure in each block being taken from Zech.

The following values of p result—

$$\begin{aligned}
\log p_1 &= 9{\cdot}7560111 \\
p_2 &\quad 9{\cdot}8960884 \\
p_3 &\quad 9{\cdot}9322424 \\
p_4 &\quad 9{\cdot}9493080 \\
p_5 &\quad 9{\cdot}9593536 \\
p_6 &\quad 9{\cdot}9660080 \\
p_7 &\quad 9{\cdot}9707556 \\
p_8 &\quad 9{\cdot}9743201 \\
p_9 &\quad 9{\cdot}9770987 \\
p_{10} &\quad 9{\cdot}9793274 \\
p_{11} &\quad 9{\cdot}9811560
\end{aligned}$$

In possession of these numbers we proceed to find

$$\begin{aligned}
b_{1,1} &= \alpha b_{0,1}\, p_1 \\
b_{2,1} &= \alpha b_{1,1}\, p_2 \qquad \text{etc. etc.,}
\end{aligned}$$

then $\mathrm{D}b_{i,1}$ by

$$\begin{aligned}
(\mathrm{D}-0)b_{0,1} &= \beta^2 b_{0,1}(1-p_1) \qquad \beta^2 = \left(\frac{a}{a_1}\right)^2 \\
(\mathrm{D}-1)b_{1,1} &= 3\beta^2 b_{1,1}(1-p_2) \\
(\mathrm{D}-2)b_{2,1} &= 5\beta^2 b_{2,1}(1-p_3) \\
&\text{etc.} \qquad\qquad \text{etc.}
\end{aligned}$$

We have now to find $\mathrm{D}^{2+m}b_{0,1}$ and $\mathrm{D}^{2+m}b_{1,1}$, which do not fall into a more simple general formula for $\mathrm{D}^{2+m}b_{i+2,1}$. We therefore use

$$D^0(D^2-i^2)b_{i,1}=\beta^2[2D+1]b_{i,1} \qquad N.B.—b_{i,1} \text{ is omitted in the next lines.}$$

$$D(D^2-i^2) \quad =\beta^2[4D^2+5D+2(1-i^2)]$$

$$D^2(D^2-i^2) \quad =\beta^2[6D^3+13D^2+(12-4i^2)D+4(1-i^2)]$$

$$D^3(D^2-i^2) \quad =\beta^2[8D^4+25D^3+(38-6i^2)D^2+(28-12i^2)D+8(1-i^2)]$$

$$D^4(D^2-i^2) \quad =\beta^2[10D^5+41D^4+(88-8i^2)D^3+(104-24i^2)D^2+(64-32i^2)D+16(1-i^2)]$$

$$D^5(D^2-i^2) \quad =\beta^2[12D^6+61D^5+(170-10i^2)D^4+(280-40i^2)D^3+(272-80i^2)D^2+(144-80i^2)D+32(1-i^2)]$$

$$D^6(D^2-i^2) \quad =\beta^2[14D^7+85D^6+(292-12i^2)D^5+(620-60i^2)D^4+(832-160i^2)D^3+(688-240i^2)D^2+(320-192i^2)D+64(1-i^2)].$$

Having found $D^m b_{0,1}$ and $D^m b_{1,1}$ by these formulæ we may check the values by means of the following equations:—

$$Db_{0,1}=\alpha Db_{1,1}$$
$$D(D-1)b_{0,1}=\alpha D^2 b_{1,1}$$
$$\ldots \qquad \ldots \qquad \ldots$$
$$D(D-1)^m b_{0,1}=\alpha D^{m+1}b_{1,1}.$$

When satisfied as to the accuracy of the values obtained for these coefficients, we proceed with

$$D^{m+2}b_{2,1}=D^m[(D^2+D-0)b_{0,1}-(D-2)b_{2,1}]$$
$$D^{m+2}b_{3,1}=D^m[(D^2+D-2)b_{1,1}-(D-6)b_{3,1}]$$
$$\ldots \qquad \ldots \qquad \ldots$$
$$D^{m+i}b_{i,1}=D^m[(D^2+D-(i-1)(i-2))b_{i-2,i}-(D-i(i-1))b_{i,1}].$$

With the above sets of formulæ the following table has been computed:—

Titan-Hyperion ($\alpha=0{\cdot}8250634$).

i	$b_{i,1}$	$Db_{i,1}$	$D^2b_{i,1}$	$D^3b_{i,1}$	$D^4b_{i,1}$	$D^5b_{i,1}$	$D^6b_{i,1}$
0	2·607466	2·389585	15·74935	170·9130	2706·378	56736·48	1482263
1	1·226640	2·896245	16·19242	171·8703	2713·127	56814·88	1483538
2	0·796696	2·820306	16·91202	175·3909	2735·725	57057·92	1487413
3	0·562370	2·611792	17·39780	180·5431	2776·454	57491·06	1494094
4	0·412874	2·359360	17·54728	186·1462	2834·065	58140·99	1503924
5	0·310212	2·098800	17·36659	191·2088	2904·347	59020·83	1517334
6	0·236676	1·847254	16·90218	195·0217	2981·309	60121·47	1534702
7	0·182556	1·613119	16·21328	197·1492	3058·366	61410·81	1556265
8	0·141972	1·400109	15·35940	197·3859	3129·180	62836·30	1582006
9	0·111120	1·209330	14·39452	195·7050	3188·272	64331·31	1611631
10	0·087419	1·040391					
11	0·069063						
12	(0·05478 est.)						
13	(0·04360 est.)						

For values of $D^m b_{i,n}$ (n other than 1) the formulæ

$$n^2 D^m b_{i+1,n+2} = D^m\left[D^2 + D - \left(i + \frac{n-1}{2}\right)\left(i + \frac{n+1}{2}\right)\right] b_{i,n}$$

$$n^2 D^m b_{i-1,n+2} = D^m\left[D^2 + D - \left(i - \frac{n-1}{2}\right)\left(i - \frac{n+1}{2}\right)\right] b_{i,n}$$

will be used, firstly with $n = 1$, then $n = 3, 5, 7$, and so on.

The following empirical formula, which is only valid for large values of α, is curious—

$$D^6 b_{1,1} = D^6 b_{0,1} + \tfrac{1}{2}[D^4 - D^3 + D^2] b_{0,1}.$$

In the case of Titan-Hyperion this shows no deviation to 7 places of figures.

It will be remarked that from first to last no term of an infinite series is computed; the nearest approach to an infinite series is the continued fraction for p_{11}.

It is well to have checks of an entirely independent nature on the values of the coefficients. Suitable checks will be found in the following formulæ, where indeed use is made of infinite series, but these series are so convergent that even with α so large as 0·825 . . . three or four terms are sufficient to ensure accuracy in the eighth significant figure.

These formulæ have been derived by Gaussian transformations of the ordinary hypergometric series.

Put

$$\alpha = \sin\theta, \qquad \alpha_1 = \cos\theta, \qquad \frac{\alpha}{\alpha_1} = \tan\theta,$$

$$\cos^2\frac{\theta}{2} = \frac{1+\alpha_1}{2},$$

and to abbreviate

$$\sin\theta \tan^2\theta = \mathrm{st} \quad \text{and} \quad \tan^4\frac{\theta}{2} = x\,;$$

then

$$b_{0,1} = \frac{2}{\cos^2\frac{\theta}{2}} F(\tfrac{1}{2}, \tfrac{1}{2}, 1.\, x)$$

$$(D+1) b_{0,1} = 1.2 \frac{\cos^2\frac{\theta}{2}}{\alpha_1^2} F(-\tfrac{1}{2}, -\tfrac{1}{2}, 1.\, x)$$

$$(D+1)(D+3) b_{0,1} = 1.3.2 \frac{\cos^6\frac{\theta}{2}}{\alpha_1^4} F(-\tfrac{3}{2}, -\tfrac{3}{2}, 1.\, x)$$

$$(D+1)(D+3)(D+5)b_{0,1} = 1.3.5.2\frac{\cos^{10}\frac{\theta}{2}}{a_1^6}F(-\tfrac{5}{2}, -\tfrac{5}{2}, 1.x)$$

$$(D+1)(D+3)\ldots(D+7)b_{0,1} = 1.3.5.7.2\frac{\cos^{14}\frac{\theta}{2}}{a_1^8}F(-\tfrac{7}{2}, -\tfrac{7}{2}, 1.x)$$

. . . . . .

$$(D-1)b_{1,1} = \frac{1.3}{2.4}.2\,\mathrm{st}\,\frac{1}{\cos^2\frac{\theta}{2}}F(-\tfrac{1}{2}, \tfrac{1}{2}, 2.x)$$

$$(D+2)(D-1)b_{1,1} = \frac{3.5}{2.4}.2\,\mathrm{st}\,\frac{\cos^2\frac{\theta}{2}}{a_1^2}F(-\tfrac{3}{2}, -\tfrac{1}{2}, 2.x)$$

$$(D+2)(D+4)(D-1)b_{1,1} = \frac{5.7}{2.4}2\,\mathrm{st}\,\frac{\cos^6\frac{\theta}{2}}{a_1^4}F(-\tfrac{5}{2}, -\tfrac{3}{2}, 2.x)$$

. . . . . .

$$(D-2)(D-4)b_{2,1} = 1.3.\frac{1.3.5.7}{2.4.6.8}2\,(\mathrm{st})^2\,\frac{1}{\cos^2\frac{\theta}{2}}F(-\tfrac{3}{2}, \tfrac{1}{2}, 3.x)$$

$$(D+3)(D-2)(D-4)b_{2,1} = 1.3.\frac{3.5.7.9}{2.4.6.8}2\,(\mathrm{st})^2\,\frac{\cos^2\frac{\theta}{2}}{a_1^2}F(-\tfrac{5}{2}, -\tfrac{1}{2}, 3.x)$$

$$(D+3)(D+5)(D-2)(D-4)b_{2,1} = 1.\frac{5.7.9.11}{2.4.6.8}2\,(\mathrm{st})^2\,\frac{\cos^6\frac{\theta}{2}}{a_1^4}F(-\tfrac{7}{2}, -\tfrac{3}{2}, 3.x)$$

. . . . . .

$$(D-3)(D-5)(D-7)b_{3,1} = 1.3.5.\frac{1.3.5.7.9.11}{2.4.6.8.10.12}2\,(\mathrm{st})^3\,\frac{1}{\cos^2\frac{\theta}{2}}F(-\tfrac{5}{2}, \tfrac{1}{2}, 4.x)$$

$$(D+4)(D-3)(D-5)(D-7)b_{3,1} = 1.3.5.\frac{3.5.\ldots.13}{2.4.\ldots.12}2\,(\mathrm{st})^3\,\frac{\cos^2\frac{\theta}{2}}{a_1^2}F(-\tfrac{7}{2}, -\tfrac{1}{2}, 4.x)$$

$$(D+4)(D+6)(D-3)\ldots(D-7)b_{3,1} = 1.5\,\frac{5.7.\ldots.15}{2.4.\ldots.12}2\,(\mathrm{st})^3\,\frac{\cos^6\frac{\theta}{2}}{a_1^4}F(-\tfrac{9}{2}, -\tfrac{1}{2}, 4.x)$$

. . . . . .

$$(D-4)(D-6)(D-8)(D-10)b_{4,1} = 1.3.5.7\frac{1.3.5.\ldots.15}{2.4.6\ldots16}2\,(\mathrm{st})^4\frac{1}{\cos^2\frac{\theta}{2}}F(-\tfrac{7}{2}, \tfrac{1}{2}, 5.x)$$

. . . . . .

$$(D-5)(D-7)\ldots(D-13)b_{5,1} = 1.3.5.7.9\frac{1.3.5\ldots19}{2.4.6\ldots20}\,2\,(\mathrm{st})^5\,\frac{1}{\cos^2\frac{\theta}{2}}F(-\tfrac{9}{2},\tfrac{1}{2},6.x)$$

. . . . . .

$$(D-6)(D-8)\ldots(D-16)b_{6,1} = 1.3.5\ldots11\,\frac{1.3.5\ldots23}{2.4.6\ldots24}\,2\,(\mathrm{st})^6\,\frac{1}{\cos^2\frac{\theta}{2}}F(-\tfrac{11}{2},\tfrac{1}{2},7.x)$$

etc. etc. etc.

We also have by direct differentiation of $b_{0,1}$

$$Db_{0,1} = \tan\theta\,\tan\frac{\theta}{2}\left[b_{0,1} + \frac{\tan^2\frac{\theta}{2}}{\cos\theta}\,F(\tfrac{1}{2},\tfrac{1}{2},2.x)\right].$$

The formula for $b_{0,1}$ is only inserted in the above table for sake of symmetry. It can be further transformed, as is well known, but the process used for its further transformation will not apply to the other formulæ.

Special formulæ such as $(D+2)(D-1)\,b_{1,1} = (D+3)(D-2)\,b_{3,1}$ also exist.

If we apply some of these formulæ to the case of Titan-Hyperion we have

$$\theta = 55^\circ\cdot594922$$

$$\log\tan^2\theta = 0\cdot32881604. \quad \log\tan\frac{\theta}{2} = 9\cdot72196186. \quad \log\cos^2\frac{\theta}{2} = 9\cdot89349547$$

$$(D-1)b_{1,1} = \frac{3}{4}\sin\theta\tan^2\theta\,\frac{1}{\cos^2\frac{\theta}{2}}\left(1 - \frac{1}{8}x - \frac{1}{64}x^2 - \frac{5}{1024}x^3 - \ldots\right)$$

$$\begin{aligned}\log\text{ series} &= 9\cdot99574456\\ \text{,, factors} &= 0\cdot22686915\\ (D-1)b_{1,1} &= 0\cdot22261371 = 1\cdot6696049\end{aligned}$$

The previous calculation gives $1\cdot6696046$

Again,

$$(D^3 - 15D^2 + 71D - 105)b_{3,1} = \frac{3465}{512}\,\frac{(\sin\theta\tan^2\theta)^3}{\cos^2\frac{\theta}{2}}\left(1 - \frac{5}{16}x + \frac{9}{128}x^2 - \frac{5}{1024}x^3 - \ldots\right)$$

$$\begin{aligned}\log\text{ series} &= 9\cdot9895742\\ \text{,, factors} &= 0\cdot6728478\\ &1\cdot6624220 = 45\cdot96434\end{aligned}$$

The tables give $45\cdot96443$

With these coefficients we could proceed to form a theory of the motion of Hyperion as disturbed by Titan, but it is at least doubtful if the algebraical development of the perturbing function

converges in this case. K. F. Sundman, in considering movement in a plane (which is very nearly the case with Titan-Hyperion), shows that the development according to powers of the eccentricities is certainly divergent if

$$\alpha(1 + 2e + e^2 + \text{etc.}) > (1 - 2e_1 + e_1^2 - \tfrac{2}{3}e_1^3 + \tfrac{2}{3}e_1^4 - \text{etc.})$$

(*Ueber* die Störungen . . . . Helsingfors, 1901).

In the present case we have

$$\alpha = 0{\cdot}825$$
$$e = 0{\cdot}029$$
$$e_1 = 0{\cdot}104$$

so that the inequality is

$$0{\cdot}874 > 0{\cdot}8016$$

and the development is divergent. In this case e_1 is not really an eccentricity in the planetary sense of the word; it is a perturbation due to the action of Titan; but this probably does not invalidate Sundman's criterion.

Newcomb and H. Struve have computed several of the coefficients (*Ast. Papers*, iii., and *Poulkovo*, xi_2. pp. 267 and 283). Using H. Struve's value of α (0·8250634) and Newcomb's clear notation we have

$$k_1 = e_1 P^{0,1}_{0,1} + e_1^3 P^{0,3}_{0,1} + e_1^5 P^{0,5}_{0,1} + \text{etc.}$$
$$P^{0,1}_{0,1} = \tfrac{1}{2}(7 + D)b_{3,1} = +3{\cdot}274193.$$
$$P^{0,3}_{0,1} = \tfrac{1}{16}(-358 - 37D + 10D^2 + D^3)b_{3,1} = +3{\cdot}534766.$$
$$P^{0,5}_{0,1} = \tfrac{1}{384}(16072 + 286D - 942D^2 + 15D^3 + 17D^4 + D^5)b_{3,1} = +262{\cdot}4885$$

$$k_2 = e_1^2 P^{0,2}_{0,2} + e_1^4 P^{0,4}_{0,2} + \text{etc.}$$
$$P^{0,2}_{0,2} = \tfrac{1}{8}(202 + 29D + D^2)b_{6,1} = +14{\cdot}78513.$$
$$P^{0,4}_{0,2} = \tfrac{1}{96}(-41572 - 5360D + 113D^2 + 34D^3 + D^4)b_{6,1} = -112{\cdot}1.$$

The value of $P^{0,4}_{0,2}$ is derived from Newcomb's formulæ; those of Le Verrier give -5062 as the coefficient of $Db_{6,1}$.

$$k_3 = e_1^3 P^{0,3}_{0,3} + \text{etc.}$$
$$P^{0,3}_{0,3} = \tfrac{1}{48}(9846 + 1415D + 66D^2 + D^3)b_{9,1} = +82{\cdot}31306.$$
. . . . . .
$$P_0 = \tfrac{1}{2}(D + D^2)b_{0,1} = +9{\cdot}069466.$$
$$P_1 = \tfrac{1}{2}(7 + D)(-36 + D + D^2)b_{3,1} = \tfrac{1}{2}(-63 - 2D + 8D^2 + D^3)b_{3,1} = -139{\cdot}5363.$$

N.B. $\tfrac{1}{2}(b_{i-1,3} + b_{i+1,3}) = (D^2 + D - i^2)b_{i,1}$.

$$P_2 = \tfrac{1}{8}(202 + 29D + D^2)(-36 + D + D^2)b_{6,1} = \tfrac{1}{8}(-7272 - 842D + 195D^2 + 30D^3 + D^4)b_{6,1} = +1106{\cdot}424.$$

It will be noticed that the same coefficients repeat themselves in the last two quantities, but they are now operated on by (D^2+D-36).

Lastly, the values of the coefficients may be required for a slightly different value of a, say $a+\Delta a$. If so, the figures given for a will be useful. Let us represent any coefficient $D^m b_{i,n}$ by $f(a)$, then by Taylor's series

$$f(a+\Delta a)=f(a)+\frac{\Delta a}{a}a\frac{d}{da}f(a)+\tfrac{1}{2}\left(\frac{\Delta a}{a}\right)^2 a^2\frac{d^2}{da^2}f(a)+\text{etc.}$$

In this replace the operator $a\frac{d}{da}$ by D which gives

$$a^2\frac{d^2}{da^2}=D(D-1)$$

$$a^3\frac{d^3}{da^3}=D(D-1)(D-2),$$

etc. etc.

and Taylor's series becomes

$$f(a+\Delta a)=f(a)+\log\left(1+\frac{\Delta a}{a}\right)Df(a)+\tfrac{1}{2}\left[\log\left(1+\frac{\Delta a}{a}\right)\right]^2 D^2 f(a)+\text{etc}$$

If $\frac{\Delta a}{a}$ is so small that its square is negligible we have

$$\log\left(1+\frac{\Delta a}{a}\right)=\frac{\Delta a}{a}.$$

If the Newcomb coefficient $D^m b_{i,n}$ be put in place of $f(a)$ we have putting l for $\log\left(1+\frac{\Delta a}{a}\right)$,

$$\overbrace{D^m b_{i,n}}^{\text{For } a+\Delta a} = \overbrace{D^m b_{i,n}+lD^{m+1}b_{i,n}+2!\,l^2D^{m+2}b_{i,n}+3!\,l^3D^{m+3}b_{i,n}+\text{etc.},}^{\text{For } a}$$

which enables us to pass readily from the coefficients calculated for a to those for $a+\Delta a$.

Johannesburg, S.A.:
26th March 1909.

Notes added by Frank Robbins.

The two equations for $b_{0,1}$ may be derived from equations (4) and (7) respectively as given by Verhulst (*loc. cit.*) if it is remembered that his D is the complete elliptic integral of the first class

$$\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - c^2 \sin^2 \phi}}$$

and his b is the complementary modulus $\sqrt{1 - c^2}$.

The second and more accurate formula may be slightly simplified—it easily becomes

$$b_{0,1} = \frac{3^2}{\{1 + \alpha_1^{\frac{1}{2}} + \sqrt[4]{8(\alpha_1^{\frac{1}{2}} + \alpha_1^{\frac{3}{2}})}\}^2} \{1 + 2q^{16} + 2q^{64} + \ldots .\}^2;$$

but even so, it is somewhat troublesome, and the first formula would probably be adopted by most computers. It is certainly quicker even if a special value of N_1 is computed instead of being interpolated from the table, page 649.

Computation of N_1.

The exact value of N_1 is $\log_{10} (1 + 2q^4 + 2q^{16} + 2q^{36} + \ldots .)^2$
or $\log_{10} (1 + 4q^4 + 4q^8 + \ldots .)$.

If α is not greater than 0·85 it will be sufficient to stop at $4q^4$, and then it is easy to see from the equations

$$\alpha^2 + \alpha_1^2 = 1 \qquad \frac{l}{2} = \frac{1 - \alpha_1^{\frac{1}{2}}}{2(1 + \alpha_1^{\frac{1}{2}})}$$

$$q = \frac{l}{2} + 2\left(\frac{l}{2}\right)^5 + 15\left(\frac{l}{2}\right)^9 + \ldots .$$

that $N_1 = m \log_e (1 + 4q^4 + 4q^8 + \ldots .)$
or $N_1 = 4mq^4$ very nearly

$$N_1 = 4m \left\{ \left(\frac{l}{2}\right)^4 + 8\left(\frac{l}{2}\right)^8 \right\} \qquad m \text{ being } 0{\cdot}4342945.$$

This is a very convenient working formula, and it can be computed for any given value of α in less than ten minutes. Zech's addition table serves admirably here; the argument being the *difference of the logarithms* of the two terms within the brackets. This *difference* can be found mentally from the log of the first term $\left(\frac{l}{2}\right)^4$ without the log $8\left(\frac{l}{2}\right)^8$ being known. It is merely necessary to add 0·903 0900 to the mantissa of log $\left(\frac{l}{2}\right)^4$ and to

subtract the sum from the characteristic with changed sign. This is perhaps quite a minor point, but still it will be of interest to computers, inasmuch as it saves half the figures (say 20 figures if using 7 fig. logs), and with every figure saved a possible error is avoided.

For the Gaussian relations of Section II. the reference is to *Disquisitiones circa seriem infinitam*: Werke, vol. iii. p. 130, Göttingen, 1876.

N_1 *expressed in Units of the Eighth Decimal Place.*

α	0	1	2	3	4	5	6	7	8	9
0·4	3	3	4	5	6	7	9	10	13	15
0·5	18	22	26	31	37	45	53	63	75	89
0·60	105	107	108	110	112	114	116	118	120	122
·61	124	126	128	130	132	135	137	139	141	144
·62	146	149	151	154	156	159	161	164	167	170
·63	172	175	178	181	184	187	190	193	197	200
·64	203	207	210	214	217	221	224	228	232	236
·65	239	243	247	251	256	260	264	268	273	277
·66	282	287	291	296	301	306	311	316	321	326
·67	332	337	343	348	354	360	366	372	378	384
·68	390	397	403	410	416	423	430	437	444	452
·69	459	466	474	482	490	498	506	514	523	531
·70	540	549	558	567	576	586	595	605	615	625
·71	635	645	656	667	678	689	700	712	723	735
·72	747	759	772	785	798	811	824	837	851	865
·73	879	894	909	924	939	954	970	986	1002	1019
·74	1036	1053	1070	1088	1106	1124	1143	1162	1181	1201
·75	1221	1241	1262	1283	1304	1326	1348	1370	1393	1416
·76	1440	1464	1488	1513	1539	1565	1591	1618	1645	1672
·77	1700	1729	1758	1788	1818	1849	1880	1912	1944	1977
·78	2011	2045	2080	2115	2151	2188	2225	2264	2302	2342
·79	2382	2423	2464	2507	2550	2594	2639	2684	2731	2778
·80	2826	2875	2926	2976	3028	3081	3135	3190	3246	3303
·81	3361	3420	3481	3542	3605	3669	3734	3800	3868	3937
·82	4007	4079	4152	4226	4302	4380	4459	4539	4622	4705
·83	4791	4878	4967	5058	5151	5245	5342	5440	5540	5643
0·84	5747	5854	5963	6074	6188	6304	6422	6543	6667	6793

α	$\log p_1$ 9·698 9700	$\log p_2$ 9·875 0613	$\log p_3$ 9·920 8187	$\log p_4$ 9·942 0081	$\log p_5$ 9·954 2425	$\log p_6$ 9·962 2114	$\log p_7$ 9·967 8153	$\log p_8$ 9·971 9713
	+	+	+	+	+	+	+	+
0·00	0	0	0	0	0	0	0	0
·02	217	72	36	22	14	10	8	6
·04	869	290	145	87	58	42	31	24
·06	1958	653	327	196	131	94	71	55
·08	3484	1162	582	349	233	167	125	98
·10	5452	1819	910	546	364	260	195	152
·12	7867	2627	1315	790	527	377	283	220
·14	10733	3585	1796	1079	720	515	386	301
·16	14056	4698	2354	1415	944	675	507	394
·18	17843	5968	2992	1799	1201	859	645	502
·20	22105	7400	3712	2233	1491	1067	801	624
·22	26848	8996	4516	2717	1816	1299	976	760
·24	32086	10761	5407	3255	2176	1557	1170	911
·26	37829	12701	6386	3847	2572	1842	1384	1078
·28	44093	14822	7460	4497	3009	2155	1619	1262
·30	50893	17130	8630	5207	3485	2498	1878	1463
·32	58245	19632	9902	5979	4004	2871	2159	1683
·34	66170	22336	11279	6815	4568	3276	2465	1922
·36	74687	25253	12767	7724	5179	3717	2798	2182
·38	83821	28388	14374	8703	5842	4195	3159	2465
·40	93598	31757	16102	9760	6556	4710	3549	2770
·42	104048	35372	17963	10900	7328	5269	3972	3103
·44	115200	39245	19963	12128	8161	5872	4429	3461
·46	127094	43392	22111	13451	9060	6523	4923	3849
·48	139769	47832	24420	14876	10031	7229	5460	4271
·50	153269	52583	26899	16410	11078	7990	6039	4726
·52	167645	57666	29562	18064	12208	8813	6666	5220
·54	182955	63109	32426	19847	13432	9707	7348	5759
·56	199262	68939	35507	21772	14754	10674	8088	6343
·58	216637	75186	38823	23850	16186	11723	8891	6978
·60	235166	81889	42400	26102	17743	12867	9768	7673
·62	254942	89090	46262	28542	19434	14112	10725	8433
·64	276075	96838	50442	31194	21279	15474	11774	9266
·66	298692	105191	54974	34082	23294	16966	12926	10184
·68	322939	114214	59900	37236	25503	18606	14195	11196
·70	348993	123989	65273	40694	27935	20418	15601	12321
·72	377063	134611	71154	44500	30623	22427	17163	13574
·74	407397	146196	77615	48705	33606	24666	18911	14979
·76	440299	158883	84749	53377	36938	27176	20877	16564
·78	476145	172847	92669	58598	40680	30007	23102	18363
·80	515418	188318	101526	64481	44922	33232	25648	20428
·82	558714	205572	111502	71156	49766	36933	28580	22816
·84	606855	225003	122861	78827	55370	41241	32011	25622
·86	660920	247122	135942	87740	61932	46316	36074	28958

Errata in Mr. Hinks' Tables for computing Standard Co-ordinates, Memoirs of R.A.S., vol. lvii. p. 153.

Arg. 248 *for* 6243 *read* 6232.
„ 249 „ 6222 „ 6202.
„ 255 „ 6014 „ 6017.